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# Stationary isothermic surfaces and some characterizations of the hyperplane \*

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## 1 Introduction

This is based on the author's recent work with R. Magnanini [MS2, MS3]. Let  $\Omega$  be a domain in  $\mathbb{R}^N$  with  $N \geq 3$ , and let  $u = u(x, t)$  be the unique bounded solution of the following problem for the heat equation:

$$\partial_t u = \Delta u \quad \text{in } \Omega \times (0, +\infty), \quad (1.1)$$

$$u = 1 \quad \text{on } \partial\Omega \times (0, +\infty), \quad (1.2)$$

$$u = 0 \quad \text{on } \Omega \times \{0\}. \quad (1.3)$$

The problem we consider is to characterize the boundary  $\partial\Omega$  such that the solution  $u$  has a stationary isothermic surface, say  $\Gamma$ . A hypersurface  $\Gamma$  in  $\Omega$  is said to be a *stationary isothermic surface* of  $u$  if at each time  $t$  the solution  $u$  remains constant on  $\Gamma$  ( a constant depending on  $t$  ). Examples we easily notice are isoparametric hypersurfaces. Namely,  $\Gamma$  and  $\partial\Omega$  are either parallel hyperplanes, concentric spheres, or concentric spherical cylinders. This complete classification of isoparametric hypersurfaces was given by Levi-Civita [LC] and Segre [Seg].

Almost complete characterizations of the sphere have already been obtained by [MS1, MS2] with the help of Aleksandrov's sphere theorem [Alek]. In this note,

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we consider some characterizations of the hyperplane. Assume that  $\Omega$  satisfies the uniform exterior sphere condition and  $\Omega$  is given by

$$\Omega = \{ x = (x', x_N) \in \mathbb{R}^N : x_N > \varphi(x') \}, \quad (1.4)$$

where  $\varphi = \varphi(x')$  ( $x' \in \mathbb{R}^{N-1}$ ) is a continuous function on  $\mathbb{R}^{N-1}$ . We recall that  $\Omega$  satisfies the *uniform exterior sphere condition* if there exists a number  $r_0 > 0$  such that for every  $\xi \in \partial\Omega$  there exists a ball  $B_{r_0}(y)$  satisfying  $\overline{B_{r_0}(y)} \cap \overline{\Omega} = \{\xi\}$ , where  $B_{r_0}(y)$  denotes an open ball centered at  $y \in \mathbb{R}^N$  and with radius  $r_0 > 0$ . Then we have

**Theorem 1.1** ([MS3]) *Assume that there exists a stationary isothermic surface  $\Gamma \subset \Omega$ . Then, under one of the following conditions (i), (ii), and (iii),  $\partial\Omega$  must be a hyperplane.*

- (i)  $N = 3$ .
- (ii)  $N \geq 4$  and  $\varphi$  is globally Lipschitz continuous on  $\mathbb{R}^{N-1}$ .
- (iii)  $N \geq 4$  and there exists a non-empty open subset  $A$  of  $\partial\Omega$  such that on  $A$  either  $H_{\partial\Omega} \geq 0$  or  $\kappa_j \leq 0$  for all  $j = 1, \dots, N-1$ , where  $H_{\partial\Omega}$  and  $\kappa_1, \dots, \kappa_{N-1}$  are the mean curvature of  $\partial\Omega$  and the principal curvatures of  $\partial\Omega$ , respectively, with respect to the upward normal vector to  $\partial\Omega$ .

**Remark.** When  $N = 2$ , this problem is easy. Since the curvature of the curve  $\partial\Omega$  is constant from (2.3) in Lemma 2.1 in Section 2 of this note, we see that  $\partial\Omega$  must be a straight line.

## 2 Outline of the proof of Theorem 1.1

In this section we give an outline of the proof. For the details, see [MS2, MS3]. Let  $d = d(x)$  be the distance function defined by

$$d(x) = \text{dist}(x, \partial\Omega), \quad x \in \Omega. \quad (2.1)$$

We start with a lemma.

**Lemma 2.1** *The following assertions hold:*

- (1)  $\Gamma = \{ (x', \psi(x')) \in \mathbb{R}^N : x' \in \mathbb{R}^{N-1} \}$  for some real analytic function  $\psi = \psi(x')$  ( $x' \in \mathbb{R}^{N-1}$ );
- (2) There exists a number  $R > 0$  such that  $d(x) = R$  for every  $x \in \Gamma$ ;
- (3)  $\varphi$  is real analytic and the mapping:  $\partial\Omega \ni \xi \mapsto x(\xi) \equiv \xi + R\nu(\xi) \in \Gamma$  is a diffeomorphism, where  $\nu(\xi)$  denotes the upward unit normal vector to  $\partial\Omega$  at  $\xi \in \partial\Omega$ , that is,  $\partial\Omega$  and  $\Gamma$  are parallel hypersurfaces with distance  $R$ ;
- (4) the following inequality holds:

$$-\frac{1}{r_0} \leq \kappa_j(\xi) < \frac{1}{R} \quad (j = 1, \dots, N-1) \quad \text{for every } \xi \in \partial\Omega, \quad (2.2)$$

where  $r_0 > 0$  is the radius of the uniform exterior sphere condition for  $\Omega$ ;

- (5) there exists a number  $c > 0$  satisfying

$$\prod_{j=1}^{N-1} \left( \frac{1}{R} - \kappa_j(\xi) \right) = c \quad \text{for every } \xi \in \partial\Omega. \quad (2.3)$$

*Proof.* The strong maximum principle implies that  $\frac{\partial u}{\partial x_N} < 0$ , and (1) holds. Since  $\Gamma$  is stationary isothermic, (2) follows from a result of Varadhan [Va]:

$$-\frac{1}{\sqrt{s}} \log W(x, s) \rightarrow d(x) \quad \text{as } s \rightarrow \infty,$$

where  $W(x, s) = s \int_0^\infty u(x, t) e^{-st} dt$  for  $s > 0$ . The inequality  $-\frac{1}{r_0} \leq \kappa_j(\xi)$  in (2.2) follows from the uniform exterior sphere condition for  $\Omega$ . See Lemma 2.2 of [MS2] together with Lemma 3.1 of [MS1] for the remainder.  $\square$

Let us proceed to the proof of Theorem 1.1. Set

$$\Gamma^* = \left\{ x \in \Omega : d(x) = \frac{R}{2} \right\}. \quad (2.4)$$

Denote by  $\kappa_j^*$  and  $\hat{\kappa}_j$  ( $j = 1, \dots, N-1$ ) the principal curvatures of  $\Gamma^*$  and  $\Gamma$ , respectively, with respect to the upward unit normal vectors. Then, the mean curvatures  $H_{\Gamma^*}$  and  $H_\Gamma$  of  $\Gamma^*$  and  $\Gamma$  are given by

$$H_{\Gamma^*} = \frac{1}{N-1} \sum_{j=1}^{N-1} \kappa_j^* \quad \text{and} \quad H_\Gamma = \frac{1}{N-1} \sum_{j=1}^{N-1} \hat{\kappa}_j,$$

respectively. These principal curvatures have the following relationship:

$$\kappa_j = \frac{\kappa_j^*}{1 + \frac{R}{2}\kappa_j^*} = \frac{\hat{\kappa}_j}{1 + R\hat{\kappa}_j} \quad (j = 1, \dots, N-1). \quad (2.5)$$

Let  $\mu = cR^{N-1}$ . Then, it follows from (2.3) and (2.5) that

$$\prod_{j=1}^{N-1} (1 - R\kappa_j) = \mu, \quad \prod_{j=1}^{N-1} (1 + R\hat{\kappa}_j) = \frac{1}{\mu}, \quad \text{and} \quad \prod_{j=1}^{N-1} \frac{1 - \frac{R}{2}\kappa_j^*}{1 + \frac{R}{2}\kappa_j^*} = \mu. \quad (2.6)$$

We distinguish three cases:

$$(I) \mu > 1, \quad (II) \mu < 1, \quad \text{and} \quad (III) \mu = 1.$$

Let us consider case (I) first. By the arithmetic-geometric mean inequality and the first equation of (2.6) we have

$$1 - RH_{\partial\Omega} = \frac{1}{N-1} \sum_{j=1}^{N-1} (1 - R\kappa_j) \geq \left\{ \prod_{j=1}^{N-1} (1 - R\kappa_j) \right\}^{\frac{1}{N-1}} = \mu^{\frac{1}{N-1}} > 1.$$

This shows that

$$H_{\partial\Omega} \leq -\frac{1}{R} \left( \mu^{\frac{1}{N-1}} - 1 \right) < 0. \quad (2.7)$$

Since

$$(N-1)H_{\partial\Omega} = \operatorname{div} \left( \frac{\nabla\varphi}{\sqrt{1 + |\nabla\varphi|^2}} \right) \quad \text{in } \mathbb{R}^{N-1},$$

by using the divergence theorem we get a contradiction as in the proof of Theorem 3.3 in [MS2]. In case (II), by the arithmetic-geometric mean inequality and the second equation of (2.6) we have

$$1 + RH_{\Gamma} = \frac{1}{N-1} \sum_{j=1}^{N-1} (1 + R\hat{\kappa}_j) \geq \left\{ \prod_{j=1}^{N-1} (1 + R\hat{\kappa}_j) \right\}^{\frac{1}{N-1}} = \mu^{-\frac{1}{N-1}} > 1.$$

This shows that

$$H_{\Gamma} \geq \frac{1}{R} \left( \mu^{-\frac{1}{N-1}} - 1 \right) > 0, \quad (2.8)$$

which yields a contradiction similarly.

Thus, it remains to consider case (III). By the above arguments we have

$$H_{\partial\Omega} \leq 0 \leq H_{\Gamma}. \quad (2.9)$$

Let us consider case (i) of Theorem 1.1 first. Since  $N = 3$  and  $\mu = 1$ , it follows from the third equation of (2.6) that

$$2H_{\Gamma^*} = \kappa_1^* + \kappa_2^* = 0.$$

We observe that  $\Gamma^*$  is a graph of a function on  $\mathbb{R}^2$ . Therefore, by the Bernstein's theorem for the minimal surface equation,  $\Gamma^*$  must be a hyperplane. This gives the conclusion desired. (See [GT, Giu] for the Bernstein's theorem.)

Secondly, we consider case (iii) of Theorem 1.1. We have

$$1 - RH_{\partial\Omega} = \frac{1}{N-1} \sum_{j=1}^{N-1} (1 - R\kappa_j) \geq \left\{ \prod_{j=1}^{N-1} (1 - R\kappa_j) \right\}^{\frac{1}{N-1}} = 1.$$

Hence, condition (iii) implies that

$$\kappa_j \equiv 0 \text{ on } A \text{ (} j = 1, \dots, N-1 \text{)}.$$

Then by the analyticity of  $\partial\Omega$  we get

$$\kappa_j \equiv 0 \text{ on } \partial\Omega \text{ (} j = 1, \dots, N-1 \text{)},$$

which shows that  $\partial\Omega$  must be a hyperplane.

Thus it remains to consider case (ii) of Theorem 1.1. In this case, there exists a constant  $L \geq 0$  satisfying

$$\sup_{\mathbb{R}^{N-1}} |\nabla\varphi| = L < \infty.$$

Then, it follows from (1) and (3) of Lemma 2.1 that

$$\sup_{\mathbb{R}^{N-1}} |\nabla\psi| = \sup_{\mathbb{R}^{N-1}} |\nabla\varphi| = L < \infty. \quad (2.10)$$

Hence, in view of this and (3) of Lemma 2.1, we can define a number  $K^* > 0$  by

$$K^* = \inf\{K > 0 : \psi \leq \varphi + K \text{ in } \mathbb{R}^{N-1}\}. \quad (2.11)$$

Then we have

$$\varphi \leq \psi \leq \varphi + K^* \text{ in } \mathbb{R}^{N-1}. \quad (2.12)$$

We define a real analytic function  $h$  on  $\mathbb{R}^{N-1}$  by

$$h(x') = \varphi(x') + K^* \text{ for } x' \in \mathbb{R}^{N-1}.$$

Moreover, by writing

$$M(h) = \operatorname{div} \left( \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) \text{ and } M(\psi) = \operatorname{div} \left( \frac{\nabla \psi}{\sqrt{1 + |\nabla \psi|^2}} \right),$$

from (2.9) and (2.12) we have

$$M(h) \leq 0 \leq M(\psi) \text{ and } \psi \leq h \text{ in } \mathbb{R}^{N-1}. \quad (2.13)$$

Hence, the method of sub- and super-solutions with the help of (2.10) yields that there exists  $v \in C^\infty(\mathbb{R}^{N-1})$  satisfying

$$M(v) = 0 \text{ and } \psi \leq v \leq h \text{ in } \mathbb{R}^{N-1}, \text{ and } \sup_{\mathbb{R}^{N-1}} |\nabla v| < \infty.$$

(See [MS3] for the details.) Therefore, Moser's theorem [Mo], Corollary, p. 591, implies that  $v$  is affine. We set  $\eta = \nabla v \in \mathbb{R}^{N-1}$ .

On the other hand, by the definition of  $K^*$  in (2.11), there exists a sequence  $\{z_n\}$  in  $\mathbb{R}^{N-1}$  satisfying

$$\lim_{n \rightarrow \infty} (h(z_n) - \psi(z_n)) = 0. \quad (2.14)$$

Define a sequence of functions  $\{\varphi_n\}$  by

$$\varphi_n(x') = h(x' + z_n) - h(z_n) \quad (= \varphi(x' + z_n) - \varphi(z_n)).$$

From (2.2) and (2.10) we see that all the second derivatives of  $\varphi$  are bounded in  $\mathbb{R}^{N-1}$ . Hence we can conclude that there exists a subsequence  $\{\varphi_{n'}\}$  of  $\{\varphi_n\}$  and a function  $\varphi_\infty \in C^1(\mathbb{R}^{N-1})$  such that  $\varphi_{n'} \rightarrow \varphi_\infty$  in  $C^1(\mathbb{R}^{N-1})$  as  $n' \rightarrow \infty$ . Since  $M(\varphi_n) \leq 0$  in  $\mathbb{R}^{N-1}$ , we have that  $M(\varphi_\infty) \leq 0$  in  $\mathbb{R}^{N-1}$  in the weak sense. Also, since  $0 \leq h(x' + z_{n'}) - v(x' + z_{n'})$  in  $\mathbb{R}^{N-1}$ , with the help of (2.14), letting  $n' \rightarrow \infty$  yields that

$$0 \leq \varphi_\infty(x') - \eta \cdot x' \text{ in } \mathbb{R}^{N-1}.$$

Consequently, we have

$$M(\varphi_\infty) \leq 0 = M(\eta \cdot x') \text{ and } \varphi_\infty \geq \eta \cdot x' \text{ in } \mathbb{R}^{N-1}, \text{ and } \varphi_\infty(0) = 0 = \eta \cdot 0. \quad (2.15)$$

Hence, the strong comparison principle implies that  $\varphi_\infty(x') \equiv \eta \cdot x'$  in  $\mathbb{R}^{N-1}$ . Here we have used Theorem 10.7 together with Theorem 8.19 in [GT]. Therefore we conclude that

$$\varphi(x' + z_n) - (v(x' + z_n) - K^*) \rightarrow 0 \text{ in } C^1(\mathbb{R}^{N-1}). \quad (2.16)$$

Similarly, we can obtain

$$v(x' + z_n) - \psi(x' + z_n) \rightarrow 0 \text{ in } C^1(\mathbb{R}^{N-1}). \quad (2.17)$$

Therefore, it follows from (3) of Lemma 2.1, (2.16), and (2.17) that the distance between two hyperplanes determined by two affine functions  $v$  and  $v - K^*$  must be  $R$ . Hence, since  $v - K^* \leq \varphi \leq \psi \leq v$  in  $\mathbb{R}^{N-1}$ , we conclude that

$$\psi \equiv v \text{ and } \varphi \equiv v - K^* \text{ in } \mathbb{R}^{N-1},$$

which shows that  $\partial\Omega$  is a hyperplane.  $\square$

### 3 Concluding remarks

Let us explain the relationship between Theorem 1.1 and Theorems 3.2, 3.3, and 3.4 in [MS2]. When  $\mu = 1$ , we have

$$1 + RH_\Gamma = \frac{1}{N-1} \sum_{j=1}^{N-1} (1 + R\hat{\kappa}_j) \geq \left\{ \prod_{j=1}^{N-1} (1 + R\hat{\kappa}_j) \right\}^{\frac{1}{N-1}} = 1.$$

Therefore, the assumption of Theorem 3.2 that  $H_\Gamma \leq 0$  implies that  $\hat{\kappa}_j \equiv 0$  ( $j = 1, \dots, N-1$ ). This shows that  $\Gamma$  is a hyperplane, and hence  $\partial\Omega$  must be a hyperplane. Thus, Theorem 3.2 is contained in Theorem 1.1 with its proof. In the case where  $\Omega$  is given by (1.4), Theorem 3.3 is contained in Theorem 1.1 with condition (iii). Since Theorem 3.4 does not assume the uniform exterior sphere condition for  $\Omega$ , Theorem 3.4 is independent of Theorem 1.1.

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